Fluctuations of Quantum Radiation Pressure in Dissipative Fluid

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Using the generalized Langevin equations involving the stress tensor approach, we study the dynamics of a perfectly reflecting mirror which is exposed to the electromagnetic radiation pressure by a laser beam in a fluid at finite temperature. Based on the fluctuation-dissipation theorem, the minimum uncertainty of the mirror's position measurement from both quantum and thermal noises effects including the photon counting error in the laser interferometer is obtained in the small time limit as compared with the "standard quantum limit". The result of the large time behavior of fluctuations of the mirror's velocity in a dissipative environment can be applied to the laser interferometer of the ground-based gravitational wave detector.

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It is known that quantum fluctuations of radiation pressure play an essential role in limiting the sensitivity of very precise quantum optical measurements in a laser interferometer which measures small changes in the position of the end mirror for detecting gravitational waves [1, 2]. When a mirror is exposed to a laser beam, its position is coupled to the laser intensity via quantum radiation pressure. The exerted force due to radiation pressure on the mirror can be expressed as the integral of the expectation values of the Maxwell stress tensor operators with respect to some physically realizable quantum states. While the quantum states are not the eigenstates of the stress tensor operators, the radiation force exhibits fluctuations. In [3], Wu and Ford have studied the effects of quantum fluctuations of radiation pressure on a mirror. They obtained the fluctuations in velocity and position of the mirror in terms of the mean squared fluctuations of the stress tensor of quantum electromagnetic fields that are known to suffer from a state-dependent divergence in the coincidence limit [4]. However, the integrals of these stress tensor correlation functions over space and time that can be directly linked to the mirror's velocity fluctuations are shown to be finite using the method of integration by parts by assuming a switch-on-switch-off procedure in the laser beam. In this Letter, we would like to generalize this approach by investigating the motion of a mirror driven by a fluctuating electromagnetic radiation force in a dissipative fluid at finite temperature. By calculating the velocity fluctuations of the mirror in this system, we can study how the dynamics of quantum fluctuations of radiation pressure is affected by the dissipative effects.

In classical statistical mechanics, the non-equilibrium dynamics of the Brownian motion in a stationary fluid can be described by a phenomenological but very successful approach, namely the Langevin equations. Incessant collisions of the molecules of the fluid with the Brownian body produce both resistance to the motion of the body as well as fluctuations in its trajectory. The Langevin equations phenomenologically account for these effects by introducing a term proportional to the velocity of the body that incorporates friction and dissipation as well as a stochastic force term that reflects the random kicks of the molecules in the fluid on the body that are thus related by the fluctuation-dissipation theorem . In most of applications, the stochastic noise is assumed to be completely uncorrelated and the coefficient of the friction term determines the relaxation time of the body. Notice that these Langevin equations can be straightforwardly generalized by involving the stress tensor approach to take account of the effects of the fluctuating quantum force due to radiation pressure. In this work, we will adopt this generalized Langevin equations to tackle the issue under consideration.

Here we consider a mirror of mass m which is oriented perpendicularly to the x-direction and is exposed to the electromagnetic radiation pressure in a dissipative fluid of temperature T. We assume that the incident radiation by a laser beam with a circular spot of radius R exerts a fluctuating force F on the one side of the mirror in the x direction within time interval τ. Then the corresponding Langevin equations to describe the motion of the mirror which can be treated classically in such a fluctuating environment is written as [5, 6]

$$m\frac{dv}{dt} = -\xi\, v + \eta(t) + F(t)\,, \tag{1}$$

where ξ is the friction coefficient and $\eta(t)$ is the stochastic force to account for the random kicks of the molecules of

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the fluid on the mirror. Its statistical properties can be summarized as follows:

$$\begin{aligned} \langle \eta(t) \rangle &= 0\,; \\ \langle \eta(t)\eta(t') \rangle &= 2\xi K_{\rm B} T \delta(t-t')\,, \end{aligned} \tag{2}$$

where $K_{\rm B}$ is Boltzmann's constant and the average is taken with respect to the ensemble of thermal equilibrium fluctuations of the fluid at finite temperature. The radiation force on the mirror in the x direction can be expressed by the area integral of the stress tensor

$$F = \int_A T_{xx} da\,, \tag{3}$$

where T_{ij} is the Maxwell stress tensor. When we treat the incident radiation quantum-mechanically, the expectation value of the stress tensor operator with respect to the quantum state of radiation fields is divergent due to vacuum fluctuations. We can renormalize it by replacing the stress tensor operator T_{ij} with a normal ordered one $: T_{ij} := T_{ij} - \langle T_{ij} \rangle_0$, which means to subtract out the vacuum divergence $\langle T_{ij} \rangle_0$ such that $\langle : T_{ij} : \rangle_0 = 0$. The subtracted vacuum divergence is defined in Minkowski space-time and will not affect the dynamics of the system. Now we want to investigate the dynamics of fluctuations of quantum radiation pressure by calculating the velocity fluctuations of the mirror. From the general argument related to the fluctuation-dissipation theorem, one may expect dissipation occurs with respect to the fluctuating radiation force even in vacuum [7, 8]. However, it turns out that the associated dissipative force arising from the creation of quantum radiation by a moving mirror has no significant effects under normal circumstances, and thus can be ignored as compared with the dissipative force (ξ-term in Eq.(1)) from the thermal noise that we will discuss later. We then further assume that the quantum fluctuations of incident radiation and the thermal noise of the fluid are uncorrelated since they are from different sources. Thus, the dispersion of the velocity can be written as the sum of contributions from thermal noise and quantum radiation pressure with respect to some quantum state to be specified later respectively:

$$\langle \Delta v^2 \rangle = \langle v^2 \rangle - \langle v \rangle^2 = \langle \Delta v^2 \rangle_T + \langle \Delta v^2 \rangle_{\rm RP}\,, \tag{4}$$

where

$$\langle \Delta v^2(\tau) \rangle_{\rm T} = \frac{1}{m^2} \int_0^\tau \int_0^\tau e^{-\frac{\xi}{m}(\tau - t_1)} e^{-\frac{\xi}{m}(\tau - t_2)} \left\langle \eta(t_1)\eta(t_2) \right\rangle \rangle \, dt_1 dt_2 \tag{5}$$

and

$$\langle \Delta v^2(\tau) \rangle_{\rm RP} = \frac{1}{m^2} \int_0^\tau \int_0^\tau e^{-\frac{\xi}{m}(\tau - t_1)} e^{-\frac{\xi}{m}(\tau - t_2)} \left(\langle F(t_1)\, F(t_2) \rangle \; - \; \langle F(t_1) \rangle \, \langle F(t_2) \rangle \right) dt_1 dt_2\,. \tag{6}$$

Using the noise correlation functions in Eq.(2), the first term can lead to the known result of the form

$$\langle \Delta v^2(\tau) \rangle_T = \frac{2K_{\rm B}T}{m} (1 - e^{-\frac{2\xi\tau}{m}})\,. \tag{7}$$

Notice that the force-force correlation function in Eq.(6) above can be expressed as the area integral of the correlation function of the stress tensors which is divergent in the coincidence limit where the method of regularization has to be introduced later to remove the divergence consistently [9]. However, it is more convenient to write this correlation function in terms of the normal ordered operator $: F := F - \langle F \rangle_0$ using the identity which can be justified straightforwardly:

$$\langle F(t_1)\, F(t_2) \rangle - \langle F(t_1) \rangle \, \langle F(t_2) \rangle = \langle : F(t_1) :: F(t_2) : \rangle - \langle : F(t_1) : \rangle \langle : F(t_2) : \rangle\,, \tag{8}$$

where $\langle \rangle_0$ refers to taking the expectation value with respect to the Minkowski vacuum. Here we would like to emphasize that in fact, the force-force correlation function that we try to compute here is *not* the fully normal-ordered one as you can see later. We now follow the approach of Ref.[3] by decomposing the force-force correlation function into the following three terms according to the Wick's theorem:

$$\langle : F(t_1) :: F(t_2) : \rangle = \langle : F(t_1) F(t_2) : \rangle + \langle : F(t_1) :: F(t_2) : \rangle_{\rm cross} + \langle : F(t_1) :: F(t_2) : \rangle_0\,. \tag{9}$$

The first term is the fully normal-ordered term, while the last term is the pure Minkowski vacuum term which can be ignored as long as we are only interested in the difference between a given quantum state and the vacuum state. The cross terms contain the products of the normal-ordered two point function and the vacuum two point function,

and are divergent in the coincident limit. For a single mode coherent state $|\alpha\rangle$, the expectation value of the fully normal-ordered force-force correlation can be shown to be equal to the product of the expectation value of the forces

$$\langle : F(t_1)F(t_2) :\rangle_\alpha = \langle : F(t_1) :\rangle_\alpha \langle : F(t_2) :\rangle_\alpha \,. \tag{10}$$

Then, the velocity fluctuation is now totally due to the cross term of the stress tensor two point function

$$\langle \Delta v^2 \rangle_{\rm RP} = \frac{1}{m^2}\int_0^\tau \int_0^\tau \int_A \int_A e^{-\frac{\xi}{m}(\tau-t_1)} e^{-\frac{\xi}{m}(\tau-t_2)} \langle : T_{xx}(t_1) :: T_{xx}(t_2) :\rangle_{\rm cross}\, da_1 da_2\, dt_1 dt_2 \,, \tag{11}$$

One can show that the cross term of the stress tensor two point function depends on the z component of B field only

$$\langle : T_{xx}(t_1,x_1) :: T_{xx}(t_2,x_2) :\rangle_{\rm cross} = \langle : B_z(t_1,x_1)B_z(t_2,x_2) :\rangle_\alpha \,\langle B_z(t_1,x_1)B_z(t_2,x_2)\rangle_0 \,, \tag{12}$$

for a linearly polarized plane wave normally incident to a mirror in the x direction and with the polarization vector chosen to be in the y-direction. We then decompose the z-component of the magnetic field into the mode functions as

$$B_z(x) = \sum_\omega \sqrt{\frac{2\omega}{V}} (a_\omega \,\cos(\omega x)\, e^{-i\omega t} + a_\omega^\dagger \,\cos(\omega x)\, e^{i\omega t}) \,, \tag{13}$$

where a and a^+ are the annihilation and creation operators of the quantum field for a box normalization in a volume V. The coherent state is an eigenstate of the annihilation operator

$$a_{\omega'}|\alpha\rangle = \delta_{\omega'\omega}\alpha|\alpha\rangle \,, \tag{14}$$

where α is a complex number. Then, the velocity fluctuation becomes (for details, see Ref.[3])

$$\langle \Delta v^2 \rangle_{\rm RP} = \frac{16\,\omega\,|\alpha|^2}{\pi^2 m^2\, V} \int_A \int_A J\, da_1 da_2 \,, \tag{15}$$

where

$$J = \int_0^\tau \int_0^\tau e^{-\frac{\xi}{m}(\tau-t_1)} e^{-\frac{\xi}{m}(\tau-t_2)} \frac{(t_1-t_2)^2 - a}{((t_1-t_2)^2 - b^2)^3} \cos(\omega\, t_1)\cos(\omega\, t_2) dt_1 dt_2 \,, \tag{16}$$

$$a = (z_1 - z_2)^2 - (y_1 - y_2)^2 \tag{17}$$

and

$$b^2 = (z_1 - z_2)^2 + (y_1 - y_2)^2 \,, \tag{18}$$

where ω is the angular velocity of an incident monochromatic laser beam. It is evident that the dissipative effects can reduce the mirror's velocity fluctuations that arise from the quantum radiation pressure with a typical relaxation time scale $\tau \approx m/\xi$. Notice that $\langle \Delta v^2 \rangle_{\rm RP}$ in Eg.(15) exhibits no explicit temperature dependence since radiation pressure driven by a laser beam on the mirror that we consider here in order to apply our study to the laser interferometer comes from the quantum mechanically coherent state. Thus, the temperature effects on the mirror's velocity fluctuations are only implicitly though the friction coefficient ξ.

Now it is of interest to study how the effects of dissipation could possibly reduce the mirror's position uncertainly that can be applied to the reduction of quantum noise in laser interferometer. To so do, let us consider the time τ where $\tau \ll m/\xi$ during which the velocity fluctuations from the thermal noise have not grown to become significantly large, but the time scale τ being still much larger than the intrinsic scale of the quantum source of radiation pressure, i.e., $\tau\,\omega \gg 1$. Taking the series expansion of the integral in terms of the small dimensionless parameter $\xi\tau/m$ leads to the following simple result, that is,

$$\langle \Delta v^2 \rangle_{\rm RP} \cong \langle \Delta v^2 \rangle^0_{\rm RP} \,(1 - \frac{\xi}{m}\tau) \,, \tag{19}$$

where $\langle \Delta v^2 \rangle^0_{\rm RP}$ refers to the mirror's velocity fluctuations from the quantum radiation pressure in vacuum which has been extensively studied in [3], and is of the form

$$\langle \Delta v^2 \rangle^0_{\rm RP} \cong 4\, \frac{A\,\omega\,\rho}{m^2}\,\tau \,. \tag{20}$$

ρ is the mean energy density of an incident laser beam that can be linked to the laser power P by $P = A\rho$ where A is the cross section area of the incident laser beam. Thus, the root mean squared position uncertainly of the mirror is given by

$$\Delta\, x_{\rm RP} \cong \frac{\sqrt{wP}}{m}\tau^{\frac{3}{2}}\,(1-\frac{\xi}{2m}\tau)\,. \tag{21}$$

On the other hand, there is another source of quantum noise in the laser interferometer, namely photon counting error, that result in the uncertainty of the location of the interference fringe of order [1, 2]

$$\Delta\, x_{\rm PC} \cong \frac{1}{2\,\sqrt{\omega P \tau}}\,. \tag{22}$$

Now we would like to recall that in fact a key limit to the sensitivity of such position measurement comes from the Heisenberg uncertainty principle. The reduction of the uncertainty of the position measurement is accompanied by an increase in the uncertainty in the momentum measurement. The limit to sensitivity is referred to as the "standard quantum limit", that is, in a measurement of duration τ, the minimum possible uncertainty in the determination of the position given by [1, 2]

$$(\Delta x)_{\rm SQL} = \sqrt{\frac{\tau}{m}}\,. \tag{23}$$

This is an intrinsic quantum uncertainty where the error of any position measurement can hardly lies below this quantum limit. Various works have been devoted to the use of constructive states, or squeezed light to reduce the quantum noise and therefore increase the sensitivity of the measurement even further [10, 11]. Here we may expect that the dissipative effects from the viscous fluid will also reduce the quantum noise on the mirror's position uncertainly. To know whether or not it is true, we now minimize the sum of these squared position uncertainties with respect to the power P to find a minimum uncertainty of the mirror's position [1, 2]. By doing so, the optimum laser power is found to be

$$P_{\rm opt} \cong \frac{m}{2\omega\tau^2}\,(1+\frac{\xi}{2m}\tau)\,, \tag{24}$$

with the position uncertainty

$$\Delta\, x_{\rm Q} \cong \sqrt{\frac{\tau}{m}}\,(1-\frac{\xi}{4m}\tau)\;<\;\Delta x_{\rm SQL}. \tag{25}$$

Presumably, one may conclude that the quantum noise on the mirror's position measurement can be reduced to even below the standard quantum limit due to the dissipative effects from the interaction with the viscous fluid. However, according to the fluctuation-dissipation theorem, the associated thermal noise with respect to the dissipative force will cause further mirror's position fluctuations by the amount which can be estimated from Eq.(7) in the short time limit $\xi\tau/m \ll 1$:

$$\Delta\, x_{\rm T} \cong \sqrt{\frac{K_{\rm B}T}{2m}}\,\tau\,\left(\frac{\xi\tau}{m}\right)^{\frac{1}{2}}\,. \tag{26}$$

Then, we find that the net position uncertainty, that is the root mean square of the uncertainties from both quantum and thermal noises, becomes

$$\Delta\, x_{\rm net} \cong \sqrt{\frac{\tau}{m}}\,\left[1+\left(\frac{K_{\rm B}T}{\hbar\tau^{-1}}-1\right)\frac{\xi}{4m}\tau\right]\,. \tag{27}$$

Admittedly, the net position uncertainty is larger than the standard quantum limit for $K_{\rm B}T \gg \hbar\tau^{-1}$ that is consistent with the short time limit as well as the typical energy scales of the thermal and quantum noises respectively. The above calculation in fact illustrates the fact that the fluctuation-dissipation theorem, that is to state that the thermal noise is accompanied with sources of dissipation [5, 12], indeed plays an essential role in determining the fluctuations of the system. Thus, this result provides a simple example to illustrate the fact that any macroscopic treatments in order to systematically reduce sources of noise and achieve the maximum sensitivity of the measurement in the laser interferometer must be consistently considered by including both fluctuation and dissipation effects that are correlated following the corresponding fluctuation-dissipation theorem [13].

The behavior of the mirror in the late time limit, $\xi\tau/m \gg 1$ is also of interest to us from which one can study the dynamics of fluctuations of quantum radiation pressure beyond the relaxation time. We first carry out the spatial integral over the area of the mirror exposed by a laser beam in Eq.(15) by assuming that the incident photon flux is uniform over this area, $A = \pi R^2$. We also assume that $\omega R \gg 1$ to simplify the calculations [3]. We then change the integrating variables $u = t_1 - t_2$ and $v = t_1 + t_2$, and Eq.(15) can be written as

$$\langle \Delta v^2 \rangle_{\rm RP} = \frac{16\,\omega\,|\alpha|^2}{\pi^2 m^2\, V}\, I\,, \tag{28}$$

where

$$I = \frac{\pi^2 R^2}{8}\left(\int_{-\tau}^{0} du \int_{-u}^{u+2\tau} dv + \int_{0}^{\tau} du \int_{u}^{2\tau-u} dv\right)\left(e^{-\frac{\xi}{m}v}\, u^2 \left(\frac{1}{(u^2-R^2)^2} - \frac{1}{u^4}\right)\left(\cos(\omega(v-2\tau)) + \cos(\omega\, u)\right)\right). \tag{29}$$

We do v-integral and write the result in terms of the dimensionless variable $x = \xi\, u/m$ as

$$\begin{aligned} I &= \frac{\pi^2 R^2}{4\,(\xi^2 + m^2\omega^2)} \int_0^{\frac{\xi\tau}{m}} dx\, e^{-x}\, x^2 \left(\frac{1}{(x^2 - (\frac{\xi R}{m})^2)^2} - \frac{1}{x^4}\right)\left(f_1(\frac{m\,x}{\xi}) - f_2(\frac{m\,x}{\xi})\right) \\ &= I_1 + I_2\,, \end{aligned} \tag{30}$$

where

$$f_1(\frac{m\,x}{\xi}) = (\xi^2 + m^2\omega^2)\cos(\frac{m\,\omega\,x}{\xi}) + \xi^2\cos(\frac{m\,\omega\,x}{\xi} - 2\,\omega\,\tau) - m\,\omega\,\xi\,\sin(\frac{m\,\omega\,x}{\xi} - 2\,\omega\,\tau) \tag{31}$$

and

$$f_2(\frac{m\,x}{\xi}) = e^{2\,(x - \frac{\xi\tau}{m})}\left((2\,\xi^2 + m^2\omega^2)\cos(\frac{m\,\omega\,x}{\xi}) + m\,\omega\,\xi\,\sin(\frac{m\,\omega\,x}{\xi})\right). \tag{32}$$

Notice that since this integrand exhibits the singular behavior at $x = \xi\, R/m$, one can define the above integrals in terms of the principal value. In addition, these integrals suffer from the intrinsic ultraviolet divergence from small x originally coming from the coincidence limit, and the method of integrations by parts will be implemented later to remove these divergences consistently. Now we restrict our attention to the case of the limit, $\xi\, R/m \ll 1$ which is relevant to the ground-based laser interferometer gravitational wave detector as we will study later. Then the above integrals can be further simplified by expanding the term in the integrand of I in terms of small parameter $\xi\, R/m$ as

$$\frac{1}{(x^2 - (\frac{\xi R}{m})^2)^2} - \frac{1}{x^4} \cong \frac{2}{x^6}\left(\frac{\xi\, R}{m}\right)^2 + O\left(\frac{\xi\, R}{m}\right)^3 \cdots, \tag{33}$$

and keeping the most dominant term only. Then the singular behavior at $x = \xi\, R/m$ can be removed under this approximation. Here we choose to use the following integration by parts prescription to deal with the intrinsic ultraviolet divergence for small x typically of form:

$$\int_0^\infty dx\, \frac{g(x)}{x^4}\,, \tag{34}$$

where $g(x)$ is a well-behaved function. The basic idea is to replace $1/x^4$ with $-(1/12)\,\partial_x^4 \ln[x^2]$, then do the integration by parts, and drop the divergent surface terms. It can be illustrated as follows:

$$\int \frac{g(x)}{x^4}\, dx = \frac{-1}{12}\int g(x)\,\partial_x^4 \ln[x^2]\, dx = \frac{-1}{12}\int \partial_x^4 g(x)\, \ln[x^2]\, dx\,. \tag{35}$$

The surface terms can be removed by assuming that the laser beam is switched on and off adiabatically in time [3]. This regularization method has also been employed by various authors under the name of "generalized principal value integration"[14] or "differential regularization"[15]. In the following, we will analyze the integrals I_1 and I_2 in Eq.(30) separately.

As for the integral I_1, we can rewrite this integral by using $\int_0^{\xi\tau/m} du = (\int_0^\infty - \int_{\xi\,\tau/m}^\infty)\, du$ as well as the expansion in Eq.(33) as

$$\begin{aligned} I_1 &\cong \frac{\pi^2 R^2}{2\,(\xi^2 + m^2\omega^2)}\left(\frac{\xi\, R}{m}\right)^2\left(\int_0^\infty - \int_{\frac{\xi\tau}{m}}^\infty\right) dx\,\left(\frac{e^{-x}}{x^4}\, f_1(\frac{m\,x}{\xi})\right) \\ &= I_{11} + I_{12}\,. \end{aligned} \tag{36}$$

We now apply the method of integration by parts to the first integral in Eq.(36) which exhibits the ultraviolet divergence for small x and then carry out the integral straightforwardly given by

$$\begin{aligned} I_{11} &= \frac{\pi^2 R^2 m^3\omega^3}{24\,\xi^2}\left(\frac{\xi R}{m}\right)^2\Bigg(2\arctan(\frac{m\,\omega}{\xi})\left(1-\frac{3\,\xi^2}{m^2\omega^2}-\frac{2\,\xi^2}{m^2\omega^2}\cos(2\omega\tau)+(\frac{\xi}{m\omega}-\frac{\xi^3}{m^3\omega^3})\sin(2\omega\tau)\right) \\ &\qquad +\left(\gamma+\ln(1+\frac{m^2\omega^2}{\xi^2})\right)\left(\frac{\xi^3}{m^3\omega^3}-\frac{3\,\xi}{m\omega}-(\frac{\xi}{m\omega}-\frac{\xi^3}{m^3\omega^3})\cos(2\omega\,\tau)-\frac{2\,\xi^2}{m^2\omega^2}\sin(2\omega\tau)\right)\Bigg) \\ &\cong \frac{\pi^3 R^4 m\,\omega^3}{24\xi}\left(1+O\left(\frac{\ln\frac{m\,\omega}{\xi}}{\frac{m\,\omega}{\xi}}\right)\right), \end{aligned} \tag{37}$$

where $\gamma = 0.5772157\cdots$ is Euler's constant. In the last step, we have used the fact that $m\,\omega \gg \xi$ to be consistent with two previous assumptions, namely $\omega\,R \gg 1$ and $\xi\,R/m \ll 1$ and the result of the integral can be further approximated by keeping the dominant term only in this limit . However, the second integral in Eq.(36) is expected to give the exponential decay behavior in the long time limit $\xi\tau/m \gg 1$ which is given by

$$\begin{aligned} I_{12} &\cong \frac{\pi^2 R^2 m^3\omega^3\,\xi}{2\,(\xi^2+m^2\omega^2)^2}\left(\frac{\xi R}{m}\right)^2\left(\frac{m}{\xi\,\tau}\right)^4 e^{-\frac{\xi\,\tau}{m}}\left(\left(1-\frac{\xi^2}{m^2\,\omega^2}\right)\sin(\omega\tau)-\frac{2\,\xi^3}{m^3\,\omega^3}\cos(\omega\tau)\right) \\ &\cong \frac{\pi^2 R^4 m\,\omega^3}{2\,\xi}\left(\frac{1}{\omega^4\,\tau^4}\right)e^{-\frac{\xi\,\tau}{m}}\left(\left(1-\frac{3\,\xi^2}{m^2\,\omega^2}\right)\sin(\omega\,\tau)+O\left(\frac{\xi}{m\,\omega}\right)^3\right). \end{aligned} \tag{38}$$

Substituting the results of Eq.(37) and Eq.(38) into Eq.(36), we obtain

$$I_1 \cong \frac{\pi^3 R^4 m\,\omega^3}{24\xi}\left(1+e^{-\frac{\xi\,\tau}{m}}\,\frac{12\,\sin(\omega\,\tau)}{\pi\,\omega^4\,\tau^4}\left(1-\frac{3\,\xi^2}{m^2\omega^2}\right)\right). \tag{39}$$

We then follow the similar approach as the calculation of the integral I_1 to compute the integral I_2 in the large time limit leading to

$$\begin{aligned} I_2 &\cong \frac{-\pi^2 R^2 m^3\omega^3\,\xi}{2\,(\xi^2+m^2\omega^2)^2}\left(\frac{\xi R}{m}\right)^2\left(\frac{m}{\xi\,\tau}\right)^4\left(\left(1+\frac{3\,\xi^2}{m^2\omega^2}\right)\sin(\omega\tau)+2\frac{2\,\xi^3}{m^3\omega^3}\left(\cos(\omega\,\tau)-e^{-\frac{\xi}{m}\tau}\right)\right) \\ &\cong \frac{-\pi^2 R^4 m\,\omega^3}{2\,\xi}\left(\frac{1}{\omega^4\,\tau^4}\right)e^{-\frac{\xi\,\tau}{m}}\left(\left(1+\frac{\xi^2}{m^2\,\omega^2}\right)\sin(\omega\,\tau)+O\left(\frac{\xi}{m\,\omega}\right)^3\right). \end{aligned} \tag{40}$$

We plug all results of I_1 and I_2 in Eq.(39) and Eq.(40) respectively into Eq.(28), and finally can obtain the velocity fluctuations of the mirror driven by quantum radiation pressure in the presence of a dissipative environment as follows:

$$\langle \Delta v^2\rangle_{\rm RP} = \frac{32C^2|\alpha|^2}{\pi^2 m^2}\,I = \frac{2\,\pi\,\rho\,\omega^3\,R^4}{3\,m\,\xi}\left(1-e^{-\frac{\xi\,\tau}{m}}\left(\frac{\xi}{m\,\omega}\right)^2\frac{24\,\sin(\omega\,\tau)}{\pi\,\omega^4\,\tau^4}\right), \tag{41}$$

where $\rho = \omega\,|\alpha|^2/V$. This is one of the main results of this work. Note that this result bears analogy to that of the classical Brownian motion with the relaxation time scale mainly determined by the macroscopic parameters, i.e., the ratio of the mass of the mirror to the friction coefficient of the mirror in a fluid. Beyond the relaxation time scale, the velocity fluctuations start to relax to the saturated value. Not too surprisingly, this result is strikingly contrary to the case in which the mirror's velocity fluctuations driven by quantum radiation pressure in vacuum grows linearly in time in Eq.(20).

As for an application, one can use the result of Eq.(41) to estimate the effects of dissipation on the quantum fluctuations of radiation pressure in the case of the laser interferometer, such as the LIGO interferometer. Now the role of the dissipative force is played by the mechanism of air damping due to the incessant collisions of air molecules with the mirror of a laser interferometer [1, 2, 16]. The mirror in the ground-based interferometer is of mass $m \approx 10\,\mathrm{Kg}$. The laser power is about $P \approx 60\,\mathrm{W}$ with a typical angular velocity $\omega \approx 4\times 10^{15}\,\mathrm{rad/s}$ as well as the spot size $A = \pi\,R^2 \approx 3\times 10^{-5}\,\mathrm{m}^2$. The gravitational wave detection using the laser interferometer is performed at quite low pressure, say about 10^{-6} Torr where the friction coefficient of the mirror in the air can be obtained by summing the momentum transfer between the mirror and each of the molecules of the air. In [5], the friction coefficient for the 1 Hz pendulum of mass 10 Kg at pressure below 10^{-6} Torr can be estimated as $\xi \approx 10^{-8}\,\mathrm{N-s/m}$. In this case, one can estimate typical energy dissipation of a mirror per oscillation due to quantum radiation using Eq.(4.18) in

Ref.[7] which is more than twenty orders of magnitude smaller than that of thermal noise with the numerical value of the fiction coefficient ξ above. This serves as a consistency check on the generalized Langivin equations in Eq.(1) that we start with. Then, with all numerical values of these parameters, the two main assumptions, i.e., $\omega R \gg 1$ and $\xi R/m \ll 1$ with which to obtain Eq.(41) are shown to be satisfied. The relaxation time scale for the mirror's velocity fluctuations can be estimated from $\tau_{\rm relax} \cong m/\xi \approx 10^9\,{\rm s}$ which is much larger than the typical measuring time scale for a ground-based interferometer, $\tau_o \approx 10^{-2}\,{\rm s}$. In addition, in order to understand quantitatively how the air damping effects reduce the mirror's velocity fluctuations, one can define γ as the ratio of the velocity fluctuations in the air (Eq.(41)) to that in vacuum (Eq.(20)). Using the above numerical parameters, we can estimate the time scale when the air damping effects reduce the velocity fluctuations by the amount, say 10%, i.e., $\gamma = 0.1$, which is about $\tau \approx 10^9\,\tau_{\rm relax} \approx 10^{18}\,{\rm s}$ twenty orders of magnitude larger than the measuring time scale τ_0. We then conclude that the air damping effects cannot significantly reduce the mirror's velocity fluctuations from the quantum radiation pressure within the measuring time scale in the case of the ground-based laster interferometer. On the other hand, according to the fluctuation-dissipation theorem, the associated thermal noise with respect to the air damping effects will cause further fluctuations on the mirror's velocity in Eq.(7), which have to be reduced in addition to the quantum noise from the radiation pressure in order to increase sensitivity of the measurement in the laser interferometer.

In summary, quantum fluctuations of electromagnetic radiation pressure in a dissipative environment have been studied. We consider a perfectly reflecting mirror which is exposed to the electromagnetic radiation pressure in a dissipative fluid at finite temperature. The dynamics of the mirror in a fluid of temperature T can be phenomenologically described by the Langevin equations that involve a linear friction force term as well as an uncorrelated (white) noise that satisfy the classical limit of the fluctuation-dissipation theorem. Here we use the generalized Langevin equations by adding a fluctuating quantum force term into the equations to incorporate the effects of radiation pressure using the stress tensor approach. We assume that quantum noise from the fluctuations of incident radiation and thermal noise of the fluid are uncorrelated since they are from different sources. Then, we compute the velocity fluctuations of the mirror which can be expressed as the sum of contributions from thermal noise and quantum radiation pressure respectively. The velocity fluctuations from thermal noise is obtained as the result of the classical Brownian motion. However, the calculations of the velocity fluctuations from radiation pressure involve the expectation value of stress tensor two point function which contains some state-dependent divergence in the coincidence limit. The integration by parts prescription by assuming a switch-on-switch-off procedure in laser beam is implemented to remove the ultraviolet divergence consistently. For a linearly polarized quantum coherent state normally incident to a mirror, the velocity fluctuations of the mirror are obtained analytically for both small time and large time limits. In the small time limit before the dynamics of the mirror's velocity fluctuations is dominated by thermal fluctuations, we find that although the dissipative effects from the interactions of the mirror with the molecules of the fluid can reduce quantum noise on the mirror's position measurement below the standard quantum limit, according to the fluctuation-dissipation theorem, the corresponding thermal noise with respect to the dissipative force will cause further mirror's position fluctuations that gives rise to the net position uncertainty, that is the root mean square of the uncertainties from both quantum and thermal noises, larger than the standard quantum limit. Thus, this result provides a simple example to illustrate the fact that any macroscopic treatments in order to systematically reduce sources of noise and achieve the maximum sensitivity of the measurement in the laser interferometer must be consistently considered by including both fluctuation and dissipation effects that are correlated following the corresponding fluctuation-dissipation theorem. The result of the large time behavior of the velocity fluctuations of a mirror in a dissipative environment is applied to the laser interferometer of the ground-based gravitational wave detector. The role of the dissipative force in this case is played by the air damping due to the incessant collisions of air molecules with the mirror. Our results reveal that in fact during the measuring time scale, the air damping effects cannot significantly reduce the mirror's velocity fluctuations from the quantum radiation pressure. This is our first attempt to tackle the issue of fluctuations of quantum radiation pressure in the presence of a dissipative environment. We note that this generalized Langevin equations using the stress tensor approach can be applied to more complicated case, for example, the dynamics of the mirror that undergoes the pendulum motion involving thermal noise of its suspension, while it is exposed by the laser beam[16]. The results of this work will have direct impact on the laser interferometer gravitational wave detector, and the work is underway.

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[1] C. M. Caves, Phy. Rev. Lett. **45**, 75 (1980).

[2] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[3] C.-H. Wu and L.H. Ford, Phys.Rev. D **64**, 045010 (2001).
[4] C.-H. Wu and L.H. Ford, Phys. Rev. D **60**, 104013 (1999).
[5] P. R. Saulson, Phys. Rev. D **42**, 2437 (1990).
[6] K. Jacobs, I. Tittonen, H. M. Wiseman, and S. Schiller, Phys. Rev. A **60**, 538 (1990).
[7] L. H. Ford and Alexander Vilenkin, Phys. Rev. D **25**, 2569 (1982).
[8] G. Barton, in *Cavity Quantum Electrodynamics*, edited by P. R. Berman (Academic Press, New York, 1993); P. V. Polevoi, Zh. Eksp. Teor. Fiz. **98**, 1990 (1990) [Sov. Phys. JETP **71**, 1119 (1990)]; L. S. Levitov , Europhys. Lett. **8**, 499 (1989); P. A. Maia Neto, and S. Reynaud, Phys. Rev. A **47**, 1639 (1993); V. V. Dodonov, A. B. Klimov, and V. I. Man'ko, Phys. Lett. A **142**, 511 (1989).
[9] G. Barton, J. Phys. A **24**, 991 (1991); *ibid.***24**, 5533 (1991).
[10] W. G. Unruth in *Quantum Optics, Experimental Gravitation and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983), p. 647; R. S. Bondurant and J. H. Shapiro, Phys. Rev. D **30**, 2548 (1984); M. T. Jaekel and S. Reynaud, Europhys. Lett. **13**, 301 (1990); A. Luis and L. L. Sanchez-Soto, Phys. Rev. A **45**, 8228 (1992).
[11] H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983); W.-T. Ni, Phys. Rev. A **33**, 2225 (1986); M. Ozawa, Phys. Rev. lett. **60**, 385 (1988).
[12] D.-S. Lee and D. Boyanovsky, Nucl. Phys. B **406** [FS] 631 (1993).
[13] J. Logan, J. Hough and N. Robertson, Phys. Lett. A **184**, 145 (1993); A. Gillespie and F. Raab, Phys. Lett. A **190**, 213 (1994).
[14] K. T. R. Davies and R. W. Davies, Can. J. Phys. **67**, 759 (1989); K. T. R. Davies, R. W. Davies, and G. D. White, J. Math. Phys. **31**, 1356 (1990).
[15] D. Z. Freedman, K. Johnson, J. I. Latorre, Nucl. Phys. B **371**, 353 (1992).
[16] Alex Abramovici et. al., Science **256**, 325 (1992).